Multi - exponential models of (1+1)-dimensional dilaton gravity and Toda - Liouville integrable models

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Abstract

The general properties of a class of two-dimensional dilaton gravity (DG) theories with multi-exponential potentials are studied and a subclass of these theories, in which the equations of motion reduce to Toda and Liouville equations, is treated in detail. A combination of parameters of the equations should satisfy a certain constraint that is identified and solved for the general multi-exponential model. From the constraint it follows that in DG theories the integrable Toda equations, generally, cannot appear without accompanying Liouville equations.

The most difficult problem in the two-dimensional Toda - Liouville DG is to solve the energy and momentum constraints. We discuss this problem using the simplest examples and identify the main obstacles to finding its analytic solution. Then we consider a subclass of integrable two-dimensional theories, in which scalar matter fields satisfy the Toda equations while the two-dimensional metric is trivial; the simplest case is considered in some detail, and on this example we outline how the general solution can be obtained.

We also show how the wave-like solutions of the general Toda - Liouville systems can be simply derived. In the dilaton gravity theory, these solutions describe nonlinear waves coupled to gravity as well as static states and cosmologies. For static states and cosmologies we propose and study a more general one-dimensional Toda - Liouville model typically emerging in one-dimensional reductions of higher-dimensional gravity and supergravity theories. A special attention is paid in this paper to making the analytic structure of the solutions of the Toda equations as simple and transparent as possible, with the aim to gain a better understanding of realistic theories reduced to dimensions 1+1 and 1+0 or 0+1.

1 Introduction

The theories of $(1+1)-$dimensional dilaton gravity coupled to scalar matter fields are known to be reliable models for some aspects of higher-dimensional black holes, cosmological models and waves. The connection between higher and lower dimensions was demonstrated in different contexts of gravity and string theory and, in several cases, has allowed finding the general solution or special classes of solutions in high-dimensional theories [1]. A generic example is the spherically symmetric

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[1]See, e.g., [1]-[28] for a more detailed discussion of this connection, references, and solution of some integrable two-dimensional and one-dimensional models of dilaton gravity.

gravity coupled to Abelian gauge fields and scalar matter fields. It exactly reduces to a (1+1)-dimensional dilaton gravity and can be explicitly solved if the scalar fields are constants independent of the coordinates[2]. These solutions can describe interesting physical objects – spherical static black holes and simplest cosmologies. However, when the scalar matter fields, which presumably play a significant cosmological role, are nontrivial, not many exact analytical solutions of high-dimensional theories are known[3]. Correspondingly, the two-dimensional models of DG that nontrivially couple to scalar matter are usually not integrable.

To construct integrable models of this sort one usually must apply serious approximations, in other words, deform the original two-dimensional model obtained by direct dimensional reductions of a realistic higher-dimensional theory. Nevertheless, the deformed models can qualitatively describe certain physically interesting solutions of higher-dimensional gravity or supergravity theories related to the low-energy limit of superstring theories. We note that several important four-dimensional space-times with symmetries defined by two commuting Killing vectors may also be described by two-dimensional models of dilaton gravity coupled to scalar matter. For example, cylindrical gravitational waves can be described by a $(1 + 1)-$dimensional dilaton gravity coupled to one scalar field [29]-[31], [22]. The stationary axially symmetric pure gravity ([32], [11]) is equivalent to a $(0+2)-$dimensional dilaton gravity coupled to one scalar field. Similar but more general dilaton gravity models were also obtained in string theory. Some of them can be solved by using modern mathematical methods developed in soliton theory (see e.g. [1], [2], [11], [19]). Note also that the theories in dimension 1+0 (cosmologies) and 0+1 (static states and, in particular, black holes) may be integrable in spite of the fact that their 1+1 dimensional 'parent' theory is not integrable without a deformation (see [23] and an example given in this paper).

In our previous work (see, e.g., [20] - [23] and references therein) we constructed and studied some explicitly integrable models based on the Liouville equation. Recently, we attempted to find solutions of some realistic two-dimensional dilaton gravity models (derived from higher-dimensional gravity theories by dimensional reduction) using a generalized separation of variables introduced in [21], [22]. These attempts showed that seemingly natural ansatzes for the structure of the separation, which proved a success in previously studied integrable models, do not give interesting enough solutions ('zero' approximation of a perturbation theory) in realistic nonintegrable models. Thus an investigation of more complex dilaton gravity models, which are based on the two dimensional Toda chains, was initiated in [24].

At first sight it seems that it should be not difficult to find a potential in DG theory that will give integrable Toda equations of motion. However in reality it is not as simple as that, and the Toda theory may only emerge in company with a Liouville theory (this was mentioned in footnote in ref. [24]). In fact, even the $N-$Liouville theory satisfies the same constraint. It was known to the authors of [23] and [24] since long time but the meaning of this fact was not clearly understood.

In this paper we first introduce the general **multi-exponential** DG and present the equations of motion in a form that resembles the Toda equations. In addition to the equations, in the DG theory one should satisfy two extra equations which in General Relativity are called **the energy and momentum constraints**. In the $N-$Liouville theory these constraints were explicitly solved but in the general case solving the constraints is a much more difficult problem which we discuss in Section 4.

Section 3 is devoted to the problem of reconstructing the dilaton gravity from the 'one-exponential' form of the equation of motion

$$\partial_u \partial_v \, x_m \; = \; g_m \exp \sum_n A_{mn} \, x_n \, . \tag{1}$$

[2]This is not possible for arbitrary dependence of the potentials on the scalar fields, as it will be clear in a moment.
[3]See, e.g., [8], [11], [12], [17]-[23]; a review and further references can be found in [26], [27] and [23].

This amounts to finding the matrix $\hat{a}$ satisfying the matrix equation[4] $\hat{a}^T\hat{\epsilon}\hat{a} = \hat{A}$ ($\hat{\epsilon}$ is a diagonal matrix to be introduced later). Evidently, this equation may have many solutions for a fixed matrix $\hat{A}$ (e.g., if $\hat{a}$ is a solution, then $\hat{O}\hat{a}$, where $\hat{O}^T\hat{\epsilon}\hat{O} = 1$, is also a solution). The important fact is however that **the solution is not possible for an arbitrary** symmetric matrix $\hat{A}^T = \hat{A}$. In Section 3 we establish the class of 'solvable' matrices $\hat{A}$ (satisfying the A-condition) and introduce a recursive procedure in order to find all possible solutions for any matrix satisfying the $A-$condition. In Appendix 2 we give the general solution of the A-condition for the matrix A_{mn} being the direct sum of a diagonal $L \times L$ matrix and of an arbitrary symmetric matrix (N-Liouville plus multi-exponential model).

The Cartan matrices for simple Lie groups do not satisfy the A-condition and thus **the generic DG cannot be reduced to the Toda equations**.[5] However, adding at least one Liouville equation to the Toda system (Toda - Liouville System, or TL) solves this constraint and in Section 4 we briefly introduce the simplest form of solution of TLS in the case of the $\mathcal{A}_n$ Cartan matrices. In addition to these standard solutions, we construct the wave-like solutions similar to ones earlier derived in the N-Liouville model. For these solutions the energy-momentum constraints are easily satisfied. In Appendix 3 we show that the form of the constraints for the general solution in the simplest $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ model is the same, but this does not help to solve them (this follows from the result of Appendix 2).

In Section 5 we turn to a simpler class of Toda - based DG models that can be completely solved (the energy-momentum constraints included). If we suppose that the potential V is independent of the dilaton φ (i.e., $V_\varphi = 0$), then the metric is flat (in the Weyl frame) and the constraints can be solved once we solve the Toda equations, which in this case need not be accompanied by the Liouville equations. This model is, in fact, a far going generalization of the well known CGHS model and can be solved directly and explicitly (although the properties of the solution are much, much more complex than their CGHS counterpart).

Section 6 is devoted to the investigation of realistic one-dimensional TL models of cosmologies and static states (e.g. black holes) that can be derived from higher dimensional gravity or supergravity. Models of this sort have been known for more than a decade but it seems that the need of the Toda - Liouville connection was not realized. We give here a complete treatment of this connection by a proper generalization of the A-condition.

Finally, in Section 7 we summarize our results in a more dogmatic form, emphasizing unsolved problems and possible applications to black holes, cosmologies and waves.

2 Multi - exponential model of (1+1)-dimensional dilaton gravity minimally coupled to scalar matter fields.

The effective Lagrangian of the (1+1)-dimensional dilaton gravity coupled to scalar fields ψ_n obtainable by dimensional reductions of a higher-dimensional spherically symmetric (super)gravity can usually be (locally) transformed to the form:

$$\mathcal{L}^{(2)} = \sqrt{-g}\left[\varphi R(g) + V(\varphi,\psi) + \sum_{m,n} Z_{mn}(\varphi,\psi)\, g^{ij}\, \partial_i\psi_m\, \partial_j\psi_n\right] \tag{2}$$

(see [20] - [23] for a detailed motivation and examples). In Eq.(2), $g_{ij}(x^0, x^1)$ is the (1+1)-dimensional metric with signature (-1,1), $g \equiv \det(g_{ij})$, R is the Ricci curvature of the two-

[4] We call it the A-equation.

[5] Due to the A-equation, the A_{mn} in Eq.(1) must be symmetric. When the Cartan matrices are non-symmetric, Eq.(1) depends on the symmetrized matrices (see Appendix 1).

dimensional space-time with the metric

$$ds^2 = g_{ij}\, dx^i\, dx^j\,, \quad i,j = 0,1\,. \tag{3}$$

The effective potentials V and Z_{mn} depend on the dilaton $\varphi(x^0,x^1)$ and on $N-2$ scalar fields $\psi_n(x^0,x^1)$ (we note that the matrix Z_{mn} should be negative definite to exclude the so called 'phantom' fields). They may depend on other parameters characterizing the parent higher-dimensional theory (e.g., on charges introduced in solving the equations for the Abelian fields). Here we consider the 'minimal' kinetic terms with diagonal and constant Z-potentials, $Z_{mn}(\varphi,\psi) = \delta_{mn} Z_n$[6]. This approximation excludes the important class of the sigma - model - like scalar matter discussed, e.g., in [28]; such models can be integrable if $V \equiv 0$ and the $Z_{mn}(\varphi,\psi)$ satisfy certain rather stringent conditions. In (2) we also used the Weyl transformation to eliminate the gradient term for the dilaton. To simplify derivations, we write the equations of motion in the light-cone metric,

$$ds^2 = -4f(u,v)\, du\, dv\,.$$

By first varying the Lagrangian in generic coordinates and then passing to the light-cone coordinates we obtain the equations of motion (Z_n are constants!)

$$\partial_u \partial_v \varphi + f\, V(\varphi,\psi) = 0, \tag{4}$$

$$f \partial_i(\partial_i \varphi / f) \; = \sum Z_n\, (\partial_i \psi_n)^2\,, \qquad i = u,v\,. \tag{5}$$

$$2Z_n\, \partial_u \partial_v\, \psi_n + f\, V_{\psi_n}(\varphi,\psi) = 0\,, \tag{6}$$

$$\partial_u \partial_v \ln|f| + f V_\varphi(\varphi,\psi) = 0\,, \tag{7}$$

where $V_\varphi \equiv \partial_\varphi V$, $V_{\psi_n} \equiv \partial_{\psi_n} V$. These equations are not independent. Actually, (7) follows from (4) $-$ (6). Alternatively, if (4), (5), and (7) are satisfied, one of the equations (6) is also satisfied. Note that the equations may have the solution with $\psi_n = \psi_n^{(0)} = \text{const}$ only if $V_{\psi_n}(\varphi, \psi_n^{(0)}) \equiv 0$.

The higher-dimensional origin of the Lagrangian (2) suggests that the potential is the sum of exponentials of linear combinations of the scalar fields and of the dilaton φ. [7] In our previous work [23] we studied the constrained Liouville model, in which the system of equations of motion (4), (6) and (7) is equivalent to the system of independent Liouville equations for the linear combinations of fields $q_n \equiv F + q_n^{(0)}$, where $F \equiv \ln|f|$. The easily derived solutions of these equations should satisfy the constraints (5), which was the most difficult part of the problem. The solution of the whole problem revealed an interesting structure of the moduli space of the solutions that allowed us to easily identify static, cosmological and wave-like solutions and effectively embed these essentially one-dimensional (in some broad sense) solutions into the set of all two-dimensional solutions and study their analytic and asymptotic properties.

Here we propose a natural generalization of the Liouville model to the model in which the fields are described by the Toda equations (or by nonintegrable deformations of them). To demonstrate that the model shares many properties with the Liouville one and to simplify a transition from the integrable models to nonintegrable theories we suggest a different representation of the Toda solutions which is not directly related to their group - theoretical background (Section 4).

Consider the theory defined by the Lagrangian (2) with the potential

$$V = \sum_{n=1}^{N} 2g_n \exp q_n^{(0)}\,, \qquad q_n^{(0)} \equiv a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}\,, \tag{8}$$

[6] In Section 6 we add to the Lagrangian (2) with minimal ψ-coupling certain fields σ whose Z-potentials depend on φ and ψ but are independent of σ, see, e.g., [23].

[7] Actually, the potential V usually contains terms non exponentially depending on φ (e.g., linear in φ), and then the exponentiation of φ is only an approximation, see the discussion in [23].

and with $Z_{mn}(\varphi, \psi) = -\delta_{mn}$. In what follows we also use the fields

$$q_n \equiv F + q_n^{(0)} \equiv \sum_{m=1}^{N} \psi_m a_{mn} \,, \tag{9}$$

where $\psi_1 + \psi_2 \equiv \ln|f| \equiv F$, $\psi_1 - \psi_2 \equiv \varphi$ and hence $a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$.

Rewriting the equations of motion in terms of ψ_n we find that Eqs. (4) - (7) are equivalent to N equations of motion for N functions ψ_n (ε is the sign of the metric f),

$$\partial_u \partial_v \psi_n = \varepsilon \sum_{m=1}^{N} \epsilon_n a_{nm} g_m \exp\left(q_m\right) \quad (\epsilon_1 = -1, \;\; \epsilon_n = +1 \text{ if } n \geq 2)\,, \tag{10}$$

and two constraints,

$$C_i \equiv \partial_i^2 \varphi + \sum_{n=1}^{N} \epsilon_n (\partial_i \psi_n)^2 = 0, \quad\;\; i = u, v\,. \tag{11}$$

With arbitrary parameters a_{nm}, these equations of motion are not integrable. However, as proposed in [16] - [18], [20] [23], Eqs.(10) are integrable and the constraints (11) can be solved if the N-component vectors $v_n \equiv (a_{mn})$ are pseudo-orthogonal.

Now, consider more general nondegenerate matrices a_{mn} and define the new scalar fields x_n:

$$x_n \equiv \sum_{m=1}^{N} a_{nm}^{-1} \epsilon_m \psi_m \,, \qquad\quad \psi_n \equiv \sum_{m=1}^{N} \epsilon_n a_{nm} x_m \,. \tag{12}$$

In terms of these fields, Eqs.(10) read as

$$\partial_u \partial_v x_m \equiv \varepsilon g_m \exp(\sum_{k,n=1}^{N} \epsilon_n a_{nm} a_{nk} x_k\,) \equiv \varepsilon g_m \exp(\sum_{k=1}^{N} A_{mk} x_k\,)\,, \tag{13}$$

and we see that the symmetric matrix

$$\hat{A} \,\equiv\, \hat{a}^T \,\hat{\epsilon}\, \hat{a}\,, \qquad\; \epsilon_{mn} \equiv \epsilon_m \;\delta_{mn}\,, \tag{14}$$

defines the main properties of the model.

If $\hat{A}$ is a diagonal matrix we return to the N-Liouville model. If $\hat{A}$ were the Cartan matrix of a simple Lie algebra, the system (13) would coincide with the corresponding Toda system, which is integrable and can be more or less explicitly solved (see, e.g., [33], [34]). However, in Section 3 we show that the Cartan matrices of the simple Lie algebras (symmetrized when necessary) cannot be represented in the form (14). Nevertheless, a very simple extension of the Toda equations obtained by adding one or more Liouville equations can resolve this problem. In fact, a symmetric matrix A_{mn} that is the direct sum of a diagonal $L \times L$-matrix $\gamma_n^{-1}\delta_{mn}$ and of an arbitrary symmetric matrix $\bar{A}_{mn}$, can be represented in form (14) if the sum of γ_n^{-1} is a certain function of the matrix elements $\bar{A}_{mn}$. If $\bar{A}_{mn}$ is a Cartan matrix, the system (13) thus reduces to L independent Liouville (Toda $\mathcal{A}_1$) equations and the higher-rank Toda system (Toda - Liouville system, or, TLS).

The solution of TLS can be derived in several ways. The most general one is provided by the group-theoretical construction described in [33], [34]. Here, in Section 4 we outline an analytical method directly applicable to solving $\mathcal{A}_N$ TLS proposed in [24]. However, solving the equations of motion is not the whole story. Once the equations are solved, their solutions must be constrained to satisfy the zero energy-momentum conditions (11) that in terms of x_n are:

$$-\,C_i \;\equiv\; 2\sum_{n=1}^{N} \;\partial_i^2 x_n \;-\; \sum_{n,m=1}^{N} \;\partial_i x_m \; A_{mn}\, \partial_i x_n \;=\; 0\,, \quad\; i = u, v\,. \tag{15}$$

In the N-Liouville model the most difficult problem was to satisfy the constraints (15) but this problem was eventually solved. In the general nonintegrable case of an arbitrary matrix $\hat{A}$, we do not know even how to approach this problem. The Toda - Liouville case is discussed below[8].

To study the general properties of the solutions of equations (13) and of the constraints (15) we first rewrite the general equations in a form that is particularly useful for the Toda - Liouville systems. Introducing the notation

$$X_n \equiv \exp(-\frac{1}{2}A_{nn}x_n) \,, \quad \Delta_2(X) \equiv X\ \partial_u\partial_v X - \partial_u X\ \partial_v X, \quad \alpha_{mn} \equiv -2A_{mn}/A_{nn} \,, \tag{16}$$

it is easy to rewrite Eqs.(13) in the form:

$$\Delta_2(X_n) = -\frac{1}{2}\varepsilon\ g_n A_{nn} \prod_{m\neq n} X_m^{\alpha_{nm}} \,. \tag{17}$$

The multiplier $|-\frac{1}{2}\varepsilon\ g_n A_{nn}|$ can be removed by using the transformation $x_n \mapsto x_n + \delta_n$ and the final (standard) form of the equations of motion is

$$\Delta_2(X_n) = \varepsilon_n \prod_{m\neq n} X_m^{\alpha_{nm}} \,, \qquad \varepsilon_n \equiv \pm 1. \tag{18}$$

These equations are in general not integrable. However, when A_{mn} are Toda plus Liouville matrices, they simplify to integrable equations (see [33]). The Liouville part is diagonal while the Toda part is non-diagonal. For example, for the Cartan matrix of $\mathcal{A}_N$, only the near-diagonal elements of the matrix α_{mn} are nonvanishing, $\alpha_{n+1,n-1} = \alpha_{n-1,n+1} = 1$. This allows one to solve Eq.(18) for any N. The parameters α_{mn} are invariant w.r.t. the transformations $x_n \mapsto \lambda_n x_n + \delta_n$. This means that the non-symmetric Cartan matrices of $\mathcal{B}_N$, $\mathcal{C}_N$, $\mathcal{G}_2$, and $\mathcal{F}_4$ can be symmetrized while not changing the equations. In this sense, the α_{mn} are the fundamental parameters of the equations of motion. From this point of view, the characteristic property of the Cartan matrices is the simplicity of Eqs.(18) which allow one to solve them by a generalization of separation of variables. As is well known, when A_{mn} is the Cartan matrix of any simple algebra, this procedure gives the exact general solution (see [33]). In Section 4 we show how to construct the exact general solution for the $\mathcal{A}_N$ Toda system and write a convenient representation for the general solution that differs from the standard one given in [33].

Unfortunately, as we emphasized above, solving equations (18) is not sufficient for finding the solution of the whole problem. We also must solve the constraints (15), and this is a much more difficult task. In our previous papers we succeeded in solving the constraints of the N-Liouville theory. So, let us try to formulate the problem of the constraints in the Toda - Liouville case as close as possible to the N-Liouville case. First, it is not difficult to show that $\partial_v C_u = \partial_u C_v = 0$ and thus $C_u = C_u(u)$, $C_v = C_v(v)$ as in the Liouville case. To prove this one should differentiate (15) and use (13) to get rid of $\partial_u\partial_v x_m$ and $\partial_u\partial_v x_n$.

Up to now we considered an arbitrary symmetric matrix $\hat{A}$. At this point we should use a more detailed information about A_{mn} and about the structure of the solution. To see whether the constraints can be solved we first rewrite them in terms of X_n and then consider the Toda - Liouville matrices and the explicit solutions of the equations. It is not difficult to see that the constraints (15) can be written in the form ($i = u$ or $i = v$ and the prime denotes ∂_i):

$$\frac{1}{4}C_i = \sum_{n=1}^{N} \gamma_n \frac{X_n''}{X_n} + \sum_{m<n}^{N} \frac{2A_{mn}}{A_m A_n} \frac{X_m'}{X_m} \frac{X_n'}{X_n} \,, \quad \gamma_n \equiv A_n^{-1} \,. \tag{19}$$

[8] In Section 5 we introduce a simplified DG Toda model that can be completely solved, including the constraints.

The first term looks exactly as in the case of the $N-$Liouville model. However, in the Liouville case we also knew that

$$\partial_u \Big(X_n^{-1}\, \partial_v^2 X_n \Big) \,=\, 0, \quad \partial_v \Big(X_n^{-1}\, \partial_u^2 X_n \Big) \,=\, 0, \tag{20}$$

which is not true in the general case. Moreover, the first and the second terms in r.h.s. of Eq.(19) are in general not functions of a single variable (above we have only proved that in general $C_u = C_u(u)$ and $C_v = C_v(v)$).

Nevertheless, let us try to push the analogy with the Liouville case as far as possible, at least in the integrable Toda - Liouville case. Thus, suppose that the first N_1 equations are the Toda ones and the remaining $N_2 = N - N_1$ equations are the Liouville ones. This means that $A_{mn} = \tilde{A}_{mn}$ $(1 \leq m, n \leq N_1)$, where $\tilde{A}_{mn}$ is a Cartan matrix while for $N_1{+}1 \leq m, n \leq N$ we have $A_{mn} = \delta_{mn} A_n$. Then the constraints split into the Toda and the Liouville parts ($X^{'} \equiv \partial_i X$):

$$\frac{1}{4} C_i = \sum_{n=1}^{N_1} \frac{1}{A_n} \frac{X_n^{''}}{X_n} + \sum_{m<n}^{N_1} \frac{2A_{mn}}{A_m A_n} \frac{X_m^{'}}{X_m} \frac{X_n^{'}}{X_n} + \sum_{n=N_1+1}^{N} \gamma_n \frac{Y_n^{''}}{Y_n}\,. \tag{21}$$

They are significantly different: first, because the Liouville solutions Y_n for $n \geq N_1 + 1$ satisfy the second order differential equation while the Toda solutions X_n satisfy higher order ones (see Section 4). In the general $\mathcal{A}_N$ Toda case X_1 can be written as

$$X_1 = \sum_{i=1}^{N+1} a_i(u)\, b_i(v)\,, \quad X_2 = \varepsilon_1 \Delta_2(X_1), \tag{22}$$

while in the Liouville case the solution is simply the sum of two terms (see Section 4). Moreover, for the Liouville solution $Y(u,v)$ we have

$$Y^{-1}\partial_u^2 Y = \frac{a_1^{''}(u)}{a_1(u)} = \frac{a_2^{''}(u)}{a_(u)}\,, \qquad Y^{-1}\partial_v^2 Y = \frac{b_1^{''}(v)}{b_1(v)} = \frac{b_2^{''}(v)}{b_2(v)}\,, \tag{23}$$

while in the Toda case everything is much more complex.

To understand better this fact we consider the case $N_1 = 2$, $N = 3$ with $A_{mn}(1 \leq m,n \leq 2)$ being the $\mathcal{A}_2-$ Cartan matrix and $A_{3n} = \delta_{3n} A_3$. Using $A_1 = A_2 = 2$, $A_{12} = A_{21} = -1$, we find

$$\frac{1}{2} C_i \;=\; \left(\frac{X_1^{''}}{X_1} + \frac{X_2^{''}}{X_2} - \frac{X_1^{'}}{X_1} \cdot \frac{X_2^{'}}{X_2} \right) \;-\; 4 \frac{Y_3^{''}}{Y_3} \;= 0 \tag{24}$$

where $X_2 = \varepsilon_1 \Delta_2(X_1)$, $\varepsilon_1 = \pm 1$, and Y_3 is the Liouville solution (note that according to the constraint on A_{ij}, considered in next Section, we have in this case $\gamma_3 = A_3^{-1} = -2$, as can be seen from Eq.(38) below). Although we know that $Y_3^{''}/Y_3$ and C_i are functions of one variable, we do not have at the moment simple and explicit expressions for C_i. Indeed, using (22) it is not difficult to find that

$$\partial_v(X_1^{-1}\partial_u^2 X_1) == \frac{1}{2} \Big(\sum_{j=1}^{3} a_j\, b_j \Big)^{-2} \sum_{i,j} W^{'}[a_i, a_j]\; W[b_i, b_j] \;\neq\; 0\,. \tag{25}$$

So, we should first write the explicit expression for $X_2(u,v)$ in terms of a, b, and then derive the complete first term in C_i. We construct the solutions of the $\mathcal{A}_2 \bigoplus \mathcal{A}_1$ constraints in Section 4.

3 Solving $\hat{a}^T\,\hat{\epsilon}\,\hat{a}\;=\;\hat{A}$

In this section we show how to solve Eq.(14) for the matrix $\hat{a}$ in the standard DG. This is possible if and only if $\hat{A}$ satisfies certain conditions, which we explicitly derive. First, $\det\hat{A} = -\det\hat{a}^2 < 0$. This restricts the matrices $\hat{A}$ of even order but is not so severe a restriction for the odd order matrices. In fact, we can then change sign of $\hat{A}$ and of all the variables x_n and the only effect will be that all ε_n in Eq.(18) change sign. If these signs are unimportant and the two systems of equations may be considered as equivalent, the restriction does not work. As the determinants of all (symmetrized) Cartan matrices for simple groups are positive (and their eigenvalues are positive), it follows that the even-order Cartan matrices do not satisfy this restriction. A more severe restriction is related to the special structure of the matrices a_{mn} in (9). In consequence, the matrix $\hat{A}$ must satisfy one equation that we derive and explicitly solve below. In this section we consider the standard DG and in Section 6 we analyze in the same approach a somewhat different one-dimensional dilaton gravity which can be met in cosmological models.

Let us now take the general $N\times N$ matrix $\hat{a}$ of DG, with the only restriction: $a_{1n} = 1 + a_n$ and $a_{2n} = 1 - a_n$. The equations defining a_{mn} in terms of A_{mn} are

$$-\,2(a_m + a_n)\,+V_m\cdot V_n \;=\; A_{mn}\,, \qquad -4a_n \;=\; A_n - V_n^2\,, \qquad m,n = 1,...,N \tag{26}$$

where we introduced the notation $V_n \;\equiv\; (\,a_{3n},...,a_{Nn})$. As it follows from (26), our N vectors V_i in the $(N-2)-$dimensional space have $N(N-2)$ components and satisfy $N(N-1)/2$ equations:

$$(V_m - V_n)^2 \;=\; A_m + A_n - 2A_{mn}\,, \qquad m > n,\;\; m,n = 1,...,N. \tag{27}$$

These equations are invariant under $(N-2)\,(N-3)/2$ rotations of the $(N-2)-$ dimensional space and under $N-2$ translations. It follows that the vectors V_m in fact depend on $(N-2)\,(N+1)/2$ invariant parameters and the number of equations minus the number of parameters is equal to one. Therefore, one of the equations should give a relation between the parameters.

It is possible to give a more constructive approach directly utilizing the invariant equations that follow from the equations (27). Define $v_k \equiv V_k - V_1$ $(k = 2,...,N)$; then, from (27) we have:

$$v_k^2 \;\equiv\; (V_k - V_1)^2 \;=\; A_1 + A_k - 2A_{1k} \;\equiv\; \tilde{A}_{1k}\,,$$

$$(v_k - v_l)^2 \;\equiv\; \tilde{A}_{1k} \,+\, \tilde{A}_{1l} \,-\, 2v_k\cdot v_l\,, \qquad k > l;\;\;\; k,l = 2,...,N\,.$$

Thus the general invariant equations for v_k can be written:

$$v_k\cdot v_l \;=\; A_1 - A_{1k} - A_{1l} + A_{kl}\,, \qquad k \geq l\,. \tag{28}$$

As these equations are valid also for $l = k$ we have $N(N-1)/2$ equations for the same number of the invariant parameters $v_k\cdot v_l$. But, of course, there is one relation between these parameters because there exist a linear relation between $N-1$ vectors v_k in the $(N-2)-$ dimensional space. For example, v_N^2 can be expressed in terms of the remaining parameters $v_2^2,...,v_{N-1}^2$ and $v_k\cdot v_l$, $k > l$ (their number is $(N-2)(N+1)/2$, as above). As the equations for v_k express $v_k\cdot v_l$ in terms of the matrix elements A_{kl}, we thus can derive the necessary relation between A_{kl} (e.g., an expression of $A_1 \equiv A_{11}$ in terms of the remaining matrix elements).

Using the vectors v_k we can give an explicit construction of the solutions and derive the constraint on the matrix elements A_{mn}. The construction of the solution of the equations for a_{mn} can be given as follows. It is not difficult to understand that we only need to find the unit vectors,

$$\hat{v}_k \equiv v_k/|v_k| \;=\; v_k\,\tilde{A}_{1k}^{-1/2}\,, \tag{29}$$

in any fixed coordinate system in the $(N-2)-$ dimensional space. Then we can reconstruct the general solution by applying to $\hat{v}_k$ rotations and translations (i.e. choosing arbitrary a_{n1}, $n=3,...,N$). Let us introduce the temporary notation

$$c_{kl} \equiv \cos\theta_{kl} \equiv \hat{v}_k \cdot \hat{v}_l \;=\; (A_1 - A_{1k} - A_{1l} + A_{kl})\,(\tilde{A}_{1k}\,\tilde{A}_{1l})^{-1/2}. \tag{30}$$

As $v_k = (a_{3k}-a_{31},...,a_{Nk}-a_{n1}\,)$, we denote $\alpha_{nk} \equiv (a_{nk}-a_{n1})/|v_k|$ and thus $\hat{v}_k = (\alpha_{3k},...,\alpha_{Nk})$. Choosing the coordinate system in which $\hat{v}_2 = (1,0,..0)$ we see that $\alpha_{3k} = c_{k2} \equiv \cos\theta_{2k}$ and $\hat{v}_3$ can be chosen with two nonvanishing components,

$$\hat{v}_3 = (c_{23}, s_{23}, 0, ..., 0)\,, \tag{31}$$

where $s_{23} \equiv \sin\theta_{23}$ and in general $s_{kl} = \sin\theta_{kl}$. The further invariant parameters α_{nk} can be derived recursively. The vectors $\hat{v}_k, ..., \hat{v}_N$ for $k \geq 4$ are constructed as follows. We take $\alpha_{3k} = c_{2k}$, $\alpha_{nk} = 0$ if $k \leq N-2$ and $n \geq k+2$. Then

$$\hat{v}_k = (c_{2k}, \alpha_{4k}, \alpha_{5k}, ..., \alpha_{(k+1)k}, 0, 0...) \tag{32}$$

and the parameters α_{nk} can be recursively derived from the relations $(k \geq 4)$

$$\sum_{n=4}^{l+1} \alpha_{nk}\alpha_{nl} = c_{kl} - c_{k2}c_{l2}\,; \quad k > l, \qquad\qquad \sum_{n=4}^{k+1} \alpha_{nk}^2 \;=\; s_{k2}^2, \quad k \leq N-1\,. \tag{33}$$

The normalization condition for $\hat{v}_N$ (not included in the above equations),

$$\sum_{n=4}^{N} \alpha_{nN}^2 \;=\; s_{N2}^2\,, \tag{34}$$

then gives a relation between the c_{kl}'s (and thus between the A_{ij}'s).

Using this solution we can find the expression for $A_1 \equiv A_{11}$ in terms of A_{kl}. However, this derivation is rather awkward. It can be somewhat simplified if we consider simpler matrices A_{kl} for which $A_{1k} = A_{k1} = 0$, $k \neq 1$. Then one can find that the equation for A_1 is linear and thus has a unique solution. Nevertheless it is not a good idea to derive the constraint on A_{kl} in this rather indirect way. The linearity of the constraint in A_1 suggests that there exists a simple and general formula directly expressing A_1 in terms of the other elements A_{kl}.

The simplest way to find A_1 in terms of the other A_{ij} is the following: one of the vectors v_2, v_3, ..., v_N must be given by a linear combination of $N-2$ other vectors. Suppose that

$$v_2 = \sum_{p=3}^{N} v_p\, z_p\,. \tag{35}$$

Then we can find z_p in terms of A_{mn} by solving the equations

$$v_p \cdot v_2 \;=\; \sum_{q=3}^{N} (v_p \cdot v_q)\, z_q\,, \quad p = 3, ..., N\,. \tag{36}$$

The solution is given by $z_p = D_p/D$, where D is the determinant of the matrix $(v_p \cdot v_q)$, and the D_p are the determinants of the same matrix but with the $p-$th column replaced by $(v_p \cdot v_2)$.

Now it is clear that the expression of v_2^2 in terms of the solution of (36),

$$v_2^2 \;=\; \sum_{q=3}^{N} (v_2 \cdot v_q)\; z_q \;=\; \sum_q (v_2 \cdot v_q)\,\cdot\, D_q/D, \tag{37}$$

gives us the desired constraint on A_{mn}. Using (28) we rewrite it in the form

$$(A_1 + A_2 - 2A_{12})\, D \;=\; \sum_{p=3}^{N} (A_1 + A_{p2} - A_{12} - A_{1p}\,)\, D_p\,, \tag{38}$$

where the determinants D and D_p should be expressed in terms of A_{mn}. They evidently depend on A_1 linearly and thus Eq.(38) is at most quadratic in A_1. In fact, it is just linear. To prove this it is sufficient to show that

$$\frac{dD}{dA_1} \;=\; \sum_{p=3}^{N} \frac{dD_p}{dA_1}\,. \tag{39}$$

To simplify the proof we introduce the following temporal notation[9]

$$D \equiv D(A_1) \equiv [C_3, C_4, ..., C_N], \qquad D_q \equiv D_q(A_1) \equiv [C_3, ..., C_{q-1}, C_2, C_{q-1}, ..., C_N]\,, \tag{40}$$

where C_k is the k-th column of the matrix $(v_p \cdot v_k)$, in particular, $C_2 \equiv (v_p \cdot v_2)$. To present differentiations in A_1 we additionally define the column C_1 all elements of which are equal to one. In this notation we have (taking into account the simple dependence of $v_k \cdot v_l$ on A_1, see (28)):

$$D'(A_1) = \sum_{q=3}^{N} [C_3, ..., C_{q-1}, C_1, C_{q-1}, ..., C_N], \;\;, \tag{41}$$

$$D'_q(A_1) = [C_3, ..., C_{q-1}, C_1, C_{q-1}, ..., C_N] \;+\; \sum_{r \neq q}^{N} [C_3, ..., C_{r-1}, C_1, C_{r-1}, ..., C_{q-1}, C_2, C_{q-1} C_N]\,. \tag{42}$$

Introducing the obvious notation, $D_{rq}(A_1)$, for the last determinants, we have

$$D'_q(A_1) = [C_3, ..., C_{q-1}, C_1, C_{q-1}, ..., C_N] \;+\; \sum_{r \neq q}^{N} D_{rq}(A_1)\,, \tag{43}$$

and thus (39) now has the form

$$\sum_{q=3}^{N} D'_q(A_1) = D'(A_1) \;+\; \sum_{q=3}^{N} \sum_{r \neq q}^{N} D_{rq}(A_1)\,. \tag{44}$$

But the determinant D_{rq} can be obtained from D_{qr} by an odd number of transpositions of the columns C_1, C_2 and thus $D_{qr} = -D_{rq}$, which completes the proof.

Now we can explicitly solve the constraint Eq.(38). Using the obvious relations

$$D = D(0) \;+\; A_1 D'(0), \qquad D_q = D_q(0) \;+\; A_1 D'_q(0)$$

and the given above expressions for the determinants in terms of A_{mn} one can write the general expression for A_1 in terms of the other matrix elements. We leave this as a simple exercise to the interested reader. Note only that the important case $A_{1n} = 0$ is somewhat simpler because then $(v_k \cdot v_l) - A_1 = A_{kl}$ and thus $D(0) = \det(A_{pq})$, etc. In Appendix we derive a beautiful solution of Eq.(38) for the models with symmetric $N \times N$ matrices A_{mn} having the form

$$A_{mn} = \delta_{mn} A_n\,, \quad 1 \le m, n \le L\,; \qquad A_{1n} = 0\,, \quad N \ge 2\,.$$

[9] Note that here $k, l = 2, 3, ..., N$ and $p, q, r = 3, ..., N$.

4 Solution of the $\mathcal{A}_N$ Toda system

The equations (18) for the $\mathcal{A}_N$-theory are extremely simple,

$$\Delta_2(X_n) = \varepsilon_n X_{n-1} X_{n+1}\,, \qquad X_0 \mapsto 1\,, \quad X_{N+1} \mapsto 1, \quad n = 1, ..., N, \tag{45}$$

where $\varepsilon_n^2 = 1$. As is well known, their solution can be reduced to solving just one higher-order equation for X_1 by using the relation (see [33]):

$$\Delta_2(\Delta_n(X)) = \Delta_{n-1}(X)\;\Delta_{n+1}(X)\,, \qquad \Delta_1(X) \equiv X, \quad n \geq 2\,, \tag{46}$$

where $\Delta_n(X)$ are determinants of the $n \times n$ matrices $X_{km} \equiv \partial_u^k \partial_v^m X$ $(1 \leq k, m \leq n)$. Indeed, using Eqs.(45), (46) one can prove that for $n \geq 2$

$$X_n = \Delta_n(X_1) \prod_{k=1}^{[n/2]} \varepsilon_{n+1-2k}\,, \tag{47}$$

where the square brackets denote the integer part of $n/2$. Thus the condition $X_{N+1} = 1$ gives the equation for X_1,

$$\Delta_{N+1}(X_1) = \prod_{k=1}^{[(N+1)/2]} \varepsilon_{N+2-2k} \equiv \tilde{\varepsilon}_{N+1} \;=\; \pm 1\,. \tag{48}$$

This equation looks horrible but it is known to be exactly soluble by a special separation of variables, Eq.(22). We present its solution in a form that is equivalent to the standard one [33] but is more compact and more suitable for constructing effectively one-dimensional solutions, generalizing those studied in [23].

Let us start with the Liouville ($\mathcal{A}_1$ Toda) equation $\Delta_2(X) = \tilde{\varepsilon}_2 \equiv \varepsilon_1$ (see [35], [36], [33], [23]). Calculating the derivatives of $\Delta_2(X)$ in the variables u and v it is not difficult to prove Eqs.(20). It follows that there exist some 'potentials' $\mathcal{U}(u)$, $\mathcal{V}(v)$ such that

$$\partial_u^2 X \,-\, \mathcal{U}(u)\, X \;=\; 0\,, \qquad \partial_v^2 X \,-\, \mathcal{V}(v)\, X \;=\; 0\,, \tag{49}$$

and thus X can be written in the 'separated' form given in (22) with $N = 1$ where $a_i(u)$, $b_i(v)$ $(i = 1, 2)$ are linearly independent solutions of the equations (Eq.(23) follows from this):

$$a_i''(u) \,-\, \mathcal{U}(u)\, a_i(u) \;=\; 0, \qquad b_i''(v) \,-\, \mathcal{V}(v)\, b_i(v) \;=\; 0\,. \tag{50}$$

For $i = 1$ these equations define the potentials for any choice of a_1, b_1, while a_2, b_2 then can be derived from the Wronskian first-order equations

$$W[a_1(u), a_2(u)] = w_a\,, \qquad W[b_1(v), b_2(v)] = w_b\,, \qquad w_a \cdot w_b = \varepsilon_1\,. \tag{51}$$

We have repeated this well known derivation at some length because it is applicable to the $\mathcal{A}_N$ Toda equation (48). By similar derivations it can be shown that X_1 satisfies the equations

$$\partial_u^{N+1} X + \sum_{n=0}^{N-1} \mathcal{U}_n(u)\; \partial_u^n X = 0\,, \qquad \partial_v^{N+1} X + \sum_{n=0}^{N-1} \mathcal{V}_n(v)\; \partial_v^n X = 0\,. \tag{52}$$

Thus the solution of (48) can be written in the same 'separated' form (22), where now $a_i(u)$, $b_i(v)$ $(i = 1, ..., N+1)$ satisfy the ordinary linear differential equations corresponding to (52), with the constant Wronskians normalized by the conditions (one can choose any other normalization in which the product of the two Wronskians is the same):

$$W[a_1(u), ..., a_{N+1}(u)] = w_a\,, \qquad W[b_1(v), ..., b_{N+1}(v)] = w_b\,, \qquad w_a \cdot w_b = \tilde{\varepsilon}_{N+1}\,. \tag{53}$$

The potentials $\mathcal{U}_n(u)$ $\mathcal{V}_n(v)$ can easily be expressed in terms of the arbitrary functions $a_i(u)$ and $b_i(v)$, $i = 1, ..., N$. To find the expressions one should differentiate the determinants (53) to obtain the homogeneous differential equations for $a_{N+1}(u)$, $b_{N+1}(v)$. For example, for $N = 2$:

$$\mathcal{U}_1(u) = -(a_1 a_2''' - a_1''' a_2)/W[a_1, a_2], \qquad \mathcal{U}_0(u) = (a_1' a_2''' - a_1''' a_2')/W[a_1, a_2] \ . \tag{54}$$

As an exercise, we suggest the reader to prove all these statements for $N = 2$. The key relation follows from the condition $\partial_u \Delta_3(X) = 0$:

$$\partial_v \left[\partial_v \left(\frac{X}{\partial_u X} \right) / \, \partial_v \left(\frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0 \ . \tag{55}$$

It follows that the expression in the square brackets is equal to an arbitrary function $A_0(u)$ and thus we have

$$\partial_v \left[\left(\frac{X}{\partial_u X} \right) \ + \ A_0(u) \left(\frac{\partial_u^3 X}{\partial_u X} \right) \right] = 0 \ . \tag{56}$$

Denoting the expression in the square bracket by $-A_1(u)$ and introducing the notation $\mathcal{U}_1(u) = A_1(u)/A_0(u)$ and $\mathcal{U}_0(u) = 1/A_1(u)$, we get the first of Eqs.(52) with $N = 2$. Repeating similar derivation starting with $\partial_v \Delta_3(X) = 0$ one can obtain the second of Eqs.(52).

Let us return to the general solution of Eq.(48). In fact, considering Eqs.(53) as inhomogeneous differential equations for $a_{N+1}(u)$, $b_{N+1}(v)$ with arbitrary chosen functions $a_i(u)$, $b_i(v)$ $(1 \leq i \leq N)$, it is easy to write the explicit solution of this problem:

$$a_{N+1}(u) = \sum_{i=1}^{N} a_i(u) \int_{u_0}^{u} d\bar{u} \ W_N^{-2}(\bar{u}) \ M_{N,\,i}(\bar{u}) \ . \tag{57}$$

Here $W_N \equiv W[a_1(u), ..., a_N(u)]$ is the Wronskian of N arbitrary chosen functions a_i and $M_{N,\,i}$ are the complementary minors of the last row in the Wronskian. (Replacing a by b and u by v we can find the expression for $b_{N+1}(v)$ from the same formula (57)). For the simplest $\mathcal{A}_2$-case:

$$a_3(u) = \sum_{i=1}^{2} a_i(u) \int_{u_0}^{u} \frac{d\bar{u}}{W_2^2(\bar{u})} \ M_{2,\,i}(\bar{u}) \equiv \int_{u_0}^{u} d\bar{u} \ \frac{a_1(\bar{u}) a_2(u) - a_1(u) a_2(\bar{u})}{(a_1(\bar{u}) a_2'(\bar{u}) - a_1'(\bar{u}) a_2(\bar{u}))^2} \ .$$

Thus we have found the expression for the basic solution X_1 in terms of $2N$ arbitrary chiral functions $a_i(u)$ and $b_i(v)$. To complete constructing the solution we should derive the expressions for all X_n in terms of a_i and b_i. This can be done with simple combinatorics that allows one to express X_n in terms of the n-th order minors. For example, it is easy to derive the expressions for X_2:

$$X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{i<j} W[a_i(u), a_j(u)] \ W[b_i(v), b_j(v)] \ , \tag{58}$$

which is valid for any $N \geq 1$ $(i, j = 1, ..., N + 1)$. Note that expressions for all X_n have a similar separated form with higher-order determinants (see [33]).

Our simple representation of the $\mathcal{A}_N$ Toda solution is completely equivalent to what one can find in [33] but is more convenient for treating some problems. For example, it is useful in discussing asymptotic and analytic properties of the solutions of the original physical problems. It is especially appropriate for constructing wave-like solutions of the Toda system which are similar to the wave solutions of the N-Liouville model. In fact, quite like the Liouville model, the Toda equations support the wave-like solutions. To derive them let us first identify the moduli space of the Toda solutions. Recalling the N-Liouville case, we may try to identify the moduli space with the space of the potentials $\mathcal{U}_n(u)$, $\mathcal{V}_n(v)$. Possibly, this is not the best choice and, in fact, in the

Liouville case we finally made a more useful choice suggested by the solution of the constraints. For our present purposes the choice of the potentials is as good as any other because each choice of $\mathcal{U}_n(u)$ and $\mathcal{V}_n(v)$ defines some solution and, vice versa, any solution given by the set of the functions $(a_1(u), ..., a_{N+1}(u))$, $(b_1(v), ..., b_{N+1}(v))$ satisfying the Wronskian constraints (53) defines the corresponding set of potentials $(\mathcal{U}_0(u), ..., \mathcal{U}_{N-1}(u))$, $(\mathcal{V}_0(v), ..., \mathcal{V}_{N-1}(v))$.

Now, as in the Liouville case, we may consider the reduction of the moduli space to the space of constant 'vectors' $(U_0, ..., U_{N-1})$, $(V_0, ..., V_{N-1})$. The fundamental solutions of the equations (52) with these potentials are exponentials (in the nondegenerate case): $\exp(\mu_i u)$, $\exp(\nu_i v)$. Then X_1 can be written as (for simplicity we take $f_i > 0$):

$$X_1 = \sum_{i=1}^{N+1} a_i(u) b_i(v) = \sum_{i=1}^{N+1} f_i \exp(\mu_i u) \exp(\nu_i v) = \sum_{i=1}^{N+1} \exp(\mu_i u + u_i) \exp(\nu_i v + v_i) \; , \tag{59}$$

where the parameters must satisfy the conditions (53). Calculating the determinant $\Delta_{N+1}(X_1)$ and denoting the standard Vandermonde determinants by

$$D_\mu \equiv \prod_{i>j} (\mu_i - \mu_j) \; , \qquad D_\nu \equiv \prod_{i>j} (\nu_i - \nu_j) \; ,$$

one can easily find that (53) is satisfied if

$$\sum_{i=1}^{N+1} \mu_i \; = \; \sum_{i=1}^{N+1} \nu_i = 0 \; , \qquad \prod_{i=1}^{N+1} f_i \; D_\mu \; D_\nu = \tilde{\varepsilon}_{N+1} \; . \tag{60}$$

By the way, instead of the last condition we could write the equivalent conditions (53):

$$\prod_{i=1}^{N+1} \exp u_i = w_a \; , \qquad \prod_{i=1}^{N+1} \exp v_i = w_b \; , \qquad w_a \cdot w_b = (D_\mu \; D_\nu)^{-1} \tilde{\varepsilon}_{N+1} \; , \tag{61}$$

where $\exp u_i$ and $\exp v_i$ are not necessary positive (e.g., we can make $\exp u_i$ negative by supposing that u_i has the imaginary part $i\pi$) but here we mostly consider positive f_i.

In this reduced case we may regard the space of the parameters $(\mu_i, \; \nu_i, \; u_i, \; v_i)$ as the new moduli space, in complete agreement with the Liouville case. Having the basic solution X_1 given by Eqs.(59)-(60) it is not difficult to derive X_n recursively by using (45). For illustration, consider the simplest TL theory $\mathcal{A}_1 \bigoplus \mathcal{A}_2$. Then X_2 is given by (58) and (59)-(60):

$$X_2 = \varepsilon_2 (D_\mu \; D_\nu)^{-1} \sum_{k=1}^{3} (\mu_i - \mu_j)(\nu_i - \nu_j) \; \exp(-\mu_k u - u_k) \; \exp(-\nu_k v - v_k) \; , \tag{62}$$

where (ijk) is a cyclic permutation of (123) (and thus $\mu_i + \mu_j = -\mu_k$, $\nu_i + \nu_j = -\nu_k$). The next step is to consider the constraints (24), where Y_3 is the solution of the Liouville equation with constant moduli.

Now, using Eqs.(59)-(62) for the $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ case, one can find that the constraints are equivalent to the following equations:

$$\sum_{i<j} (\mu_i - \mu_j)(\nu_j - \nu_j)[3\mu_k^2 - C_\mu] = 0 \; , \qquad \sum_{i<j} (\mu_i - \mu_j)(\nu_j - \nu_j)[3\nu_k^2 - C_\nu] = 0 \; , \tag{63}$$

$$\mu_1^2 + \mu_2^2 + \mu_1 \mu_2 = C_\mu \; , \qquad \nu_1^2 + \nu_2^2 + \nu_1 \nu_2 = C_\nu \; , \tag{64}$$

where the constants C_μ and C_ν represent the contribution of the Liouville term. Computing the sums in Eq.(63) we find that Eqs.(63) are equivalent to the relations

$$3[(\mu_1^2 + \mu_2^2 + \mu_1 \mu_2) - C_\mu] \sum \mu_i \nu_i = 0 \; , \quad 3[(\nu_1^2 + \nu_2^2 + \mu_1 \nu_2) - C_\nu] \sum \mu_i \nu_i = 0 \; , \tag{65}$$

which are satisfied as soon as Eqs.(64) are satisfied.

It is not difficult to check that the potentials $\mathcal{U}_1(u)$, $\mathcal{V}_1(v)$ for the exponential solutions are

$$\mathcal{U}_1(u) = -(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) \equiv \frac{1}{2}\sum \mu_i^2, \qquad \mathcal{V}_1(v) = -(\nu_1^2 + \nu_2^2 + \mu_1\nu_2) \equiv \frac{1}{2}\sum \nu_i^2, \tag{66}$$

and thus the constraints have an extremely simple and natural form:

$$\mathcal{U}_1 + C_\mu = 0\ , \qquad \mathcal{V}_1 + C_\nu = 0. \tag{67}$$

These constraints can easily be solved. In Appendix 2 we write the constraints for the general solutions in the $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ theory in the same form but with $\mathcal{U}_1$, C_μ depending on u and $\mathcal{V}_1$, C_ν depending on v. Unfortunately, this does not help to solve these more general equations.

5 A simple integrable model of (1+1)-dimensional dilaton gravity coupled to Toda scalar matter

Here we consider a simple DG model (first briefly discussed in [24]), in which the potential V is independent of φ, i.e., $\partial_\varphi V \equiv 0$. Supposing, as above, that the potentials Z_n are constant (we take $Z_n \equiv -1$) it easy to see that the equations for the matter fields ψ_n can be separated and solved independently from the other equations[10]. This obviously follows from the fact that the equation (7) for the metric $f(u,v)$ defines the essentially trivial metric $f = \varepsilon a'(u)b'(v)$, which can locally be transformed to $f = \varepsilon$. Thus, one has first to solve the matter equations (6) and then the dilaton equation (4), while the constraints (5) give additional relations between the dilaton and matter fields.

Let us define the potential V as a multi - exponential function of ψ_n

$$V \;=\; \varepsilon \sum_{n=1}^{N} 2g_n \, \exp \sum_{m=1}^{N}\, \psi_m \, a_{mn} \tag{68}$$

Then the matter equations (equations of motion) are

$$\partial_u \, \partial_v \, \psi_n \;=\; \varepsilon \, \sum_{m=1}^{N}\, a_{nm} \, g_m \, \exp \sum_{k=1}^{N}\, \psi_k \, a_{km} \, . \tag{69}$$

Now we can use for the description of the model the equations (11) - (14) if we set in them $\epsilon_n \equiv 1$. The equations for A_{mn} are very simple to solve and we discuss them at the end of this Section. The matter equations (13) are integrable in the Toda - Liouville case but they are also integrable in the pure Toda case, the simplest example being the $\mathcal{A}_2$ Toda theory.

When Eqs.(13) are integrable we can find the complete solution of the model, including the constraints. However, it is not necessary to suppose that Eqs.(13) are integrable to find particular solutions of all the equations. So, suppose that we have found a solution of the system (13) and show how to derive the dilaton field and solve the constraints. In fact, this is a very simple exercise.

The general solution of Eq.(4) can be written as

$$\varphi = -\varepsilon \int_0^u \int_0^v d\bar{u}\, d\bar{v}\, V[\psi(\bar{u},\bar{v})] + A(u) + B(v), \tag{70}$$

where $A(u)$, $B(v)$ are arbitrary functions. The constraints (5) in this model have the form

$$\partial_i^2 \varphi = -\sum_{n=1}^{N} (\partial_i \psi_n)^2, \quad\; i = u, v\, . \tag{71}$$

[10] In this section we slightly change notation and consider N matter fields ψ_n, $n = 1, ..., N$.

Using the equations of motion (6) we easily derive

$$\partial_i V = \varepsilon \partial_j \sum_1^N (\partial_i \psi_n)^2, \quad (i,j) = (u,v) \text{ or } (v,u)\,, \tag{72}$$

find $A(a)$, $B(b)$ in terms of ψ, and finally obtain:

$$\varphi = -\varepsilon \int_0^u \int_0^v d\bar{u} d\bar{v}\, V[\psi(\bar{u},\bar{v})] - \int_0^u d\bar{\bar{u}} \int_0^{\bar{\bar{u}}} d\bar{u}\; \Phi_u(\bar{u}) - \int_0^v d\bar{\bar{v}} \int_0^{\bar{\bar{v}}} d\bar{v}\; \Phi_v(\bar{v}) + A'(0)u + B'(0)v\,, \tag{73}$$

where we omitted the unimportant arbitrary term $A(0) + B(0) = \varphi(0,0)$ and denoted

$$\Phi_u(u) \equiv \sum_1^N (\partial_u \psi_n(u,0))^2\,, \qquad \Phi_v(v) \equiv \sum_1^N (\partial_v \psi_n(v,0))^2\,.$$

This completes finding the solution of the complete system of this model **provided that we know the general solution of Eqs.(6)**.

Now, to get integrable equations for ψ_n we take the potential (68) for which the equations (12) - (14) (with $\epsilon_n \equiv 1$) can be reduced to integrable Toda equations. Therefore, we find the explicit analytical solution for the nontrivial class of dilaton gravity minimally coupled to scalar matter fields. This model is a very complex generalization of the well studied CGHS model (with free scalar fields and with the trivial potential $V = g$). Our model has much richer geometric properties and a very complex structure of the space of its solutions. In particular, the construction of the solutions with constant moduli of the previous section is fully applicable and directly gives the generalized wave-like solutions that include the one-dimensional reductions – the static and cosmological solutions.

The general $\mathcal{A}_N$ case can be solved and studied using the results presented in this paper. The easiest case is $N = 1$ (the Liouville equation for a single ψ). The first simple but really interesting theory is the case of two scalar fields satisfying the $\mathcal{A}_2$ Toda equations. Taking, for example,

$$V = \exp(\sqrt{3}\; \psi_1 - \psi_2) + \exp(2\; \psi_2),$$

we find the simplest realization of the $\mathcal{A}_2$ Toda dilaton gravity model whose complete solution can be obtained by the use of the above derivations. An interesting question is the following: is it possible to derive such models by dimensional reduction of some 'realistic' higher-dimensional theories?

6 One-dimensional Toda – Liouville systems

Here we consider the one-dimensional multi-exponential models and, especially, those that can be reduced to integrable Toda - Liouville theories. These models can be derived either directly from the higher dimensional (super)gravity theories[11] or by further reducing the two-dimensional dilaton gravity. Having in mind the reduction of DG to dimension one we consider somewhat more general two-dimensional Lagrangians (we return to the notation of Section 2 for ψ_n)

$$\mathcal{L}^{(2)} \;=\; \sqrt{-g}\left[\varphi R(g) \;+\; V(\varphi,\psi) \;+\; \sum_{n=3}^{N_1} Z(\varphi)\, (\nabla\psi_n)^2 \;+\; \sum_{p=N_1+1}^{N} Z_p(\varphi,\psi)\, (\nabla\sigma_p)^2\right]. \tag{74}$$

[11] Various approaches to dimensional reductions and concrete low-dimensional models can be found in many papers published in the last twenty years, see, e.g., [3], [5], [10], [11], [13] - [22], [25], [28] and references therein.

The difference between σ_p and ψ_n is that $\mathcal{L}^{(2)}$ depends only on the derivatives of σ_p but not on σ_p itself. The simplest reduction of $\mathcal{L}^{(2)}$ to the one-dimensional case relevant for the description of black holes and of cosmologies can be set into the following form (see, e.g. [20]), [23]):

$$\mathcal{L}^{(1)} = \frac{1}{\bar{l}(\tau)} \Big(-\dot{\phi}\dot{F} + \sum_n \dot{\psi}_n^2 - \sum_p \bar{Z}_p(\phi,\psi)\,\dot{\sigma}_p^2 \Big) + \bar{l}(\tau)\,\varepsilon\, e^F\, \bar{V}(\phi,\psi)\,. \tag{75}$$

Here ϕ is related to the original dilaton φ by the equation $\phi'(\varphi) \equiv -Z(\varphi)$, $\bar{l}(\tau)$ is the new Lagrangian multiplier, $\bar{l}(\tau) = l(\tau)\,\phi'(\varphi)$, and

$$\bar{V}(\phi,\psi) = \frac{V(\varphi,\psi)}{\phi'(\varphi)}, \quad \bar{Z}_p(\phi,\psi) \;=\; -Z_p(\phi,\psi)\,\phi'(\varphi). \tag{76}$$

Note that τ may be either the time, t, or the space, r, coordinate and correspondingly ϵ is equal to ± 1, reflecting the sign of the metric f (see ref. [1] and, for a more detailed discussion, refs. [3] and [4]). In what follows we interpret Eq.(75) as a cosmological model and thus denote $\tau = t$. In fact, the cosmological Lagrangian of type (75)) can be derived directly from higher - dimensional supergravity theories (see, e.g. [17], [18], [20])).

Now, one can see that the equations of motion for the Lagrangian (75) can be solved w.r.t. the fields $\sigma_p(t)$. Indeed, the canonical momenta P_q corresponding to $\sigma_p(t)$ are conserved:

$$\frac{d}{dt}\,P_q \equiv \frac{d}{dt}\frac{\partial \mathcal{L}^{(1)}}{\partial \dot{\sigma}_q} = -2\frac{d}{dt}(Z_q(\varphi,\psi)\,\dot{\sigma}_q) \;=\; \frac{\partial \mathcal{L}^{(1)}}{\partial \sigma_q} \;\equiv\; 0, \quad N_1 + 1 \le q \le N\,. \tag{77}$$

We also have (ψ_1 and ψ_2 are defined in terms of φ, F, as in Section 2):

$$P_i \equiv \frac{\partial \mathcal{L}^{(1)}}{\partial \dot{\psi}_i}, \quad 1 \le i \le N_1, \tag{78}$$

and so we find the Hamiltonian corresponding to the Lagrangian (75),

$$\mathcal{H}^{(1)} \;=\; \frac{l(t)}{4}\Big\{ \sum_{i=1}^{N_1} \epsilon_i P_i^2 - \sum_{q=N_1+1}^{N} \bar{Z}_q^{-1}(\phi,\psi)\,P_q^2 + 4\varepsilon\, e^F\, \bar{V}(\phi,\psi) \Big\}, \tag{79}$$

where now the P_q are the integration constants ($\epsilon_1 = -1$, $\epsilon_n = 1$, $1 \le n \le N_1$). This means that the Lagrangian (75) can be replaced by the effective one,

$$\mathcal{L}^{(1)} \;=\; \frac{1}{\bar{l}(t)} \sum_{n=1}^{N_1} \dot{\psi}_n^2 \;+\; \bar{l}(t)\,\varepsilon\, V_{\text{eff}}(\phi,\psi) \tag{80}$$

where V_{eff} is the effective potential,

$$V_{\text{eff}}(\phi,\psi) \;=\; \varepsilon\, e^F\, \bar{V}(\phi,\psi) - \sum_{q=N_1+1}^{N} {P_q}^2 \bar{Z}_q(\phi,\psi)\,. \tag{81}$$

If this effective potential is of the exponential form (as in Section 2), we may apply our general approach developed for the treatment of the two−dimensional systems. Thus from now on we simply forget about the transformations above. The only difference is that the new potential V is the sum of two terms (denoting $N_2 \equiv N - N_1$ we call this (N_1, N_2)-model):

$$V = \sum_{i=1}^{N_1} 2g_i \exp\, q_i \;+\; \sum_{p=N_1+1}^{N} 2g_p \exp q_p^{(0)} \tag{82}$$

and thus $a_{1i} = 1 + a_i$, $a_{2i} = 1 - a_i$, $i = 1, ..., N_1$, while $a_{1p} = a_p$, $a_{2p} = -a_p$, $p = N_1 + 1, ..., N$.

When the equations of motion can be reduced to the Toda - Liouville system, we can use the one-dimensional solutions of Section 4. To find the class of the DG theories for which this is possible, we should find a_{mn} in terms of A_{mn} and solve the constraint on A_{mn} generalizing Eq.(38) As the matrix a_{mn} now has a different form, the solution of the equations $\hat{a}^T \hat{\epsilon} \hat{a} = \hat{A}$, which are equivalent to the bilinear system

$$- a_{1m} a_{1n} + a_{2m} a_{2n} + \sum_{k=3}^{N} a_{km} a_{kn} = A_{mn} , \tag{83}$$

do not coincide with those of Section 4, although the derivations will be very similar.

To illustrate the difference consider the simplest case where

$$a_{mn} = \begin{pmatrix} 1+a_1 & 1+a_2 & a_3 \\ 1-a_1 & 1-a_2 & -a_3 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

In this case, Eqs.(83) give six equations for six unknowns, a_1, a_2, a_3 and a_{31}, a_{32}, a_{33}. It is not difficult to find that the solution is given by (recall that $A_{mn} = A_{nm}$)

$$4a_1 = a_{31}^2 - A_1 , \qquad 4a_2 = a_{32}^2 - A_2 , \qquad a_{33} = \sqrt{A_3} ,$$

$$(a_{32} - a_{31}) = (A_{32} - A_{31})/\sqrt{A_3} , \qquad 2a_3 = a_{31} a_{33} - A_{31} = a_{32} a_{33} - A_{32} .$$

It is clear that one of the a_{mn} is not defined by the equations (e.g., A_{31}) and that the two expressions for a_3 give the constraint on a_{mn}:

$$(A_1 + A_2 - 2A_{12}) A_3 = (A_{23} - A_{13})^2 . \tag{84}$$

This constraint is similar to Eq.(38) and has the same origin and meaning. It is not satisfied for the $\mathcal{A}_3$ Cartan matrix but can be satisfied for the $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ Cartan matrix (with $A_{21} = A_{31} = 0$ and $A_1 = -3/2$).

Let us now consider the general system (83), which is not much more difficult to study than the 'standard' system (26). For this reason we omit details and only give a brief summary of the main differences. Eqs.(26), (27) are valid for $m, n \leq N_1$ (we denote such m, n by i, j).

The new equations are (the vectors V_m are defined as $V_m \equiv [a_{3m}, ..., a_{Nm}]$ for all m):

$$- 2a_p + V_p \cdot V_i = A_{pi} \equiv A_{ip} , \qquad V_p \cdot V_q = A_{pq} , \qquad i \leq N_1 , \; N_1 + 1 \leq p \leq N \tag{85}$$

The first system of equations ('ip-equations') is equivalent to ($k = 2, ..., N_1$):

$$V_p \cdot v_k \equiv V_p (V_k - V_1) = A_{ip} - A_{1p} , \quad -2a_p + V_p \cdot V_1 = A_{1p} .$$

Thus we get $N(N-1)/2$ equations for $N-1$ vectors v_k, V_p in the $(N-2)$-dimensional space:

$$v_k \cdot v_l = A_1 + A_{kl} - A_{1k} - A_{1l} , \quad k, l = 2, ..., N_1 , \tag{86}$$

$$v_k \cdot V_p \equiv A_{kp} - A_{1p} , \quad v_p \cdot V_q = A_{pq} , \quad k = 2, .., N_1 , \quad p = N_1 + 1, ..., N , \tag{87}$$

while the other equations define a_m in terms of these vectors and of the matrix elements A_{mn}:

$$4a_i = V_i^2 - A_i , \qquad 2a_p = V_p \cdot V_1 - A_{p1} , \tag{88}$$

As we have $N-1$ vectors v_k, V_p in the $(N-2)$-dimensional space, one of them is a linear combination of the other ones, e.g.,

$$v_2 = \sum_{n=3}^{N_1} x_n v_n + \sum_{p=N_1+1}^{N} x_p V_p. \tag{89}$$

It follows that the matrix elements A_{mn} should satisfy one constraint that can be obtained by first deriving x_n, x_p and then writing the expression for v_2^2 in terms of A_{mn} (see the quite similar derivation in Section 3)[12].

To actually derive a_{mn} in terms of A_{mn} we should solve th equations for v_k, V_p in a manner described in Section 3, Eqs.(29)-(34) (solving the equations in a fixed coordinate system, then using rotational invariance for the unit vectors $\hat{v}_k$, $\hat{V}_p$, etc.). A simplest nontrivial example is given by the case $N_1 = N_2 = 2$. We leave it as an exercise to the reader.

Note in conclusion, that many examples of the one-dimensional Toda and Toda - Liouville equations related to the equations considered in this and in the previous sections were derived by reductions of higher dimensional theories for description of some black holes and cosmologies (see, e.g. [37] - [40], [13] - [15], [25]). We plan to discuss applications of our formal approach and results in a forthcoming paper.

7 Conclusion

Let us briefly summarize the main results and possible applications. We introduced a simple and compact formulation of the general (1+1)-dimensional dilaton gravity with multi-exponential potentials and derived the conditions allowing to find its explicit solutions in terms of the Toda theory. The simplest class of theories satisfying these conditions is the Toda - Liouville theory. In Section 5 (see also [24]), we show that the models with the potentials independent of the dilaton φ can be explicitly solved if A_{mn} is any Cartan matrix. In this case adding the Liouville part is unnecessary. We also proposed a simple approach to solving the equations in the case of the $\mathcal{A}_N$ Toda part.

Of special interest are simple exponential solutions derived in Section 4. They explicitly unify the static (black hole) solutions, cosmological models, and waves of the Toda matter coupled to gravity. Some of these solutions can be related to cosmologies with spherical inhomogeneities or to evolving black holes but this requires special studies. Earlier we studied similar but simpler solutions in the N-Liouville theories in paper [23]. The main results of that paper, in particular the existence of nonsingular exponential solutions, are true also in the Toda - Liouville theory.

Note that the one-dimensional Toda - Liouville cosmological models were met long time ago in dimensional reductions of higher-dimensional (super)gravity theories (see, e.g., [15]). Considerations of the two-dimensional Toda - Liouville theories of this paper are equally applicable to the one-dimensional case. A preliminary discussion can be found in [24], and the detailed consideration is presented in this paper, together with a detailed presentation of the results that were only briefly described there (some of the results of Sections 2-4 were recently presented in [41]).

To include into consideration the waves one has to step up at least one dimension higher. The principal aim of the present paper was to make the first step and explore this problem in a simplest two-dimensional Toda environment. As a simple exercise (based on the results of this paper) one may consider the reductions from dimension (1+1) both to dimension (1+0) ('cosmological' reduction) and to dimension (0+1) ('static' or 'black hole' reduction) as well as the moduli space reductions to waves. One of the most interesting problems for future investigations is the connection

[12] If $N_1 = 2$ and N_2 is arbitrary, the constraint can be easily solved because the only dependence on A_1 is contained in $v_2^2 = A_1 + A_2 - 2A_{12}$ (see also the example below).

between these three objects. It was discovered in the N-Liouville theory but now we see that it can be found in a much more complex theory described by the Toda equations. It is not impossible that the connection also exists (possibly, in a weaker form) in some nonintegrable theories.

Finally, we must admit that the general problem of solving the energy an momentum constraints in the two-dimensional Toda - Liouville theory remains essentially unsolved. It is completely solved in the model of Sections 5 and in the one-dimensional model of Section 6 (as well as for the effectively one-dimensional solutions of Section 4). However, in the general two-dimensional Toda - Liouville models we cannot analytically satisfy the constraints for the general solution. To analytically solve this problem we should probably combine our analytic approach with the group-theoretical considerations of [33], [34].

8 Appendix

8.1 Cartan matrices and α_{mn}

For all Cartan matrices $A_{nn} = 2$. For all Cartan matrices, except $\mathcal{G}_2$ and $\mathcal{F}_4$,

$$A_{(n-1)n} = A_{n(n-1)} = -1, \qquad 2 \le n \le N-1\,.$$

For the Cartan matrices of $\mathcal{A}_N$, $\mathcal{B}_N$, $\mathcal{C}_N$, $\mathcal{D}_N$, $\mathcal{E}_N$ (for the last series $N = 6, 7, 8$):

$$\mathcal{A}_N, \mathcal{E}_N:\; A_{(N-1)N} = A_{N(N-1)} = -1, \quad \mathcal{D}_N:\; A_{(N-2)N} = A_{N(N-2)} = -1$$

$$\mathcal{B}_N:\; A_{(N-1)N} = -2,\, A_{N(N-1)} = -1; \quad \mathcal{C}_N:\; A_{(N-1)N} = -1,\, A_{N(N-1)} = -2.$$

For $\mathcal{E}_N$, in addition $A_{3N} = A_{N3} = -1$. The non-diagonal elements of $\mathcal{G}_2$ are $A_{12} = -1$, $A_{21} = -3$. For $\mathcal{F}_4$, all near-diagonal elements are equal to -1, except $A_{23} = -2$. We list only the nonvanishing elements, the other ones are zero.

The matrices of $\mathcal{A}_N$, $\mathcal{D}_N$, $\mathcal{E}_N$ are symmetric and thus define the symmetric α_{mn}. By $\tilde{\mathcal{B}}_N$, $\tilde{\mathcal{C}}_N$, $\tilde{\mathcal{G}}_2$, $\tilde{\mathcal{F}}_4$ we denote the symmetrized (as explained in the main text) Cartan matrices. For $\tilde{\mathcal{G}}_2$ we have $A_{12} = A_{21} = -3$, $A_{11} = 2$, $A_{22} = 6$. For $\tilde{\mathcal{F}}_4$:

$$A_{11} = A_{22} = 4, \quad A_{33} = A_{44} = 2, \quad A_{12} = A_{23} = -2, \quad A_{34} = -1\,.$$

It is easy to derive α_{mn} using Eq.(16). For $\mathcal{A}_N$: $\alpha_{nn} = -2$, $\alpha_{(n-1)n} = \alpha_{n(n-1)} = 1$ $(2 \le n \le N-1)$, and the other elements vanish. For other series and for exceptional groups, the α_{mn} are not so trivial (e.g., $\alpha_{12} = 1$, $\alpha_{21} = 2$ for $\mathcal{C}_2$) and thus our simple approach to solving the Toda equations is not directly applicable.

8.2 Constraint on general multi-exponential plus Liouville matrices A_{mn}

Let us consider symmetric $N \times N$ matrices A_{mn} satisfying the conditions

$$A_{kl} = \delta_{kl} A_k\,, \quad 1 \le l \le L\,, \quad 1 \le k \le N\,. \tag{90}$$

For convenience, let us denote $A_{L+m,L+n} \equiv B_{mn}$ for $1 \le m, n \le M \equiv N - L$, where B_{mn} is an arbitrary symmetric $M \times M$ matrix. When $L \ge 2$ the constraint equation (38) for A_{mn} has the following simple form:

$$(A_1 + A_2)\, D \;=\; A_1 \sum_{p=3}^{N} D_p\,. \tag{91}$$

As the determinants D_p are proportional to A_1 (because $(v_p \cdot v_q) \equiv A_1$) the r.h.s. is proportional to A_1^2. According to the theorem proven in Section 4, Eq.(91) is linear in A_1 and thus the quadratic

terms in both sides of the equation should cancel. It follows that it is sufficient to derive the determinant D. We can do this by induction in L. For $L = 2, 3, 4$ it is easy to find that D can be presented by the following simple formula:

$$D \equiv D(L) \;=\; \gamma_2 \Big[\delta_M (\sum_{l=3}^{L} \gamma_l + \gamma_1 + \sigma_{M-1}) \Big] \prod_{1}^{L} A_k \,, \tag{92}$$

where $\gamma_l \equiv A_l^{-1}$, $\delta_M \equiv \det \hat{B}$, and σ_{M-1} is the determinant of the $(M-1) \times (M-1)$ matrix $\hat{B}''$ having the following matrix elements

$$B''_{mn} = B_{mn} - B_{mM} - B_{Mn} + B_{MM} \,, \quad 1 \leq m, n \leq M-1 \,.$$

Direct computation of $D(L+1)$ with the aid of Eq.(92) now allows to prove that the expression (92) is valid for any L. The last step is to insert D into equation (91) which gives:

$$(A_1 + A_2)\, D \;=\; \Big\{ \Big[\delta_M (\sum_{l=3}^{L} \gamma_l + \gamma_1 + \sigma_{M-1}) \Big] + \delta_M\, \gamma_2 + A_1 \Big(\gamma_2 \sum_{l=3}^{L} \gamma_l \;+\; \sigma_{M-1} \Big) \Big\} \prod_{1}^{L} A_k \,. \tag{93}$$

Neglecting the last term that is proportional to A_1^2, we finally find that Eq.(91) gives the beautiful relation

$$\delta_M \sum_{l=1}^{L} \gamma_l + \sigma_{M-1} = 0 \,, \tag{94}$$

which is our final result.

Supposing (naturally) that $\delta_M \neq 0$ we see that $\sum_{l=1}^{L} \gamma_l = 0$ is possible if and only if $\sigma_{M-1} = 0$. Note that for $\mathcal{A}_N$ Cartan matrices $\sigma_{M-1} \neq 0$. Most probably, this is also true for all Cartan matrices of the simple groups. For the generic matrices $\hat{B}$ the condition $\sigma_{M-1} = 0$ can be solved and thus there exists a class of the matrices $\hat{B}$ for which $\sum_{l=1}^{L} \gamma_l = 0$. This means that it is possible to solve the energy and momentum constraints for any given solution of the B-system, which unfortunately, is not integrable and cannot be solved analytically.

8.3 Energy and momentum constraints for $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ theory

Here we show that the energy and momentum constraints can be written similarly to Eqs.(67) also for the general solution of the $\mathcal{A}_1 \bigoplus \mathcal{A}_2$ theory:

$$\mathcal{U}_1(u) + C_\mu(u) = 0 \;, \qquad \mathcal{V}_1(v) + C_\nu(v) = 0. \tag{95}$$

For our simplest example, the Toda solutions X_1 and X_2 are given by (22) and (58) with $N = 2$:

$$X_1 = \sum_{i=1}^{3} a_i(u)\, b_i(v) \,, \quad X_2 = \varepsilon_1 \Delta_2(X_1) = \varepsilon_1 \sum_{k=1}^{3} W_k \bar{W}_k \;, \tag{96}$$

where $W_k \equiv W[a_i(u), a_j(u)]$, $\bar{W}_k \equiv W[b_i(v), b_j(v)]$ and (ijk) is a cyclic permutation of (123). The C_u constraint (24) is equivalent to the relation

$$\tilde{C}_u \equiv X_1^{''} X_2 + X_1 X_2^{''} - X_1^{'} X_2^{'} \;-\; 4 X_1 X_2 X_L^{''} / X_L \;= 0 \,, \tag{97}$$

where the prime denotes ∂_u and X_L is any solution of the Liouville equation satisfying Eq.(49), from which it follows that

$$X_L^{''} / X_L = a^{''}(u) / a(u) \equiv \; \alpha(u). \tag{98}$$

Using Eqs.(96) - (98) it is easy to find that

$$\tilde{C}_u = \sum_{m,k} b_m \bar{W}_k \, [a_m^{''} W_k - a_m^{'} W_k^{'} + a_m W_k^{''} - 4\alpha W_k] \,. \tag{99}$$

Let us first derive the 'diagonal' part of the sum, using the definition of W_k and Eq.(53) for $N = 2$:

$$\sum_{m=k} [...] = \sum_k b_k \bar{W}_k \, [(a_i a_j^{'''} - a_i^{'''} a_j)\, a_k + (a_i^{''} W_{jk} + a_j^{''} W_{ki} + a_k^{''} W_{ij}) - 4\alpha W_k \,] \,. \tag{100}$$

Recalling the definition of $\mathcal{U}_1(u)$ (see (54)) we find that the first term in the square brackets is equal to $-\mathcal{U}_1(u) a_k W_k$. The second term in the square brackets is equal to the constant w_a, which is the Wronskian of $a_i(u)$ (see (53)). The sum $w_a \sum_k b_k \bar{W}_k$ identically vanishes because

$$\sum_k b_k \bar{W}_k = (b_1 b_2^{'} - b_1^{'} b_2) b_3 + (b_2 b_3^{'} - b_2^{'} b_3) b_1 + (b_3 b_1^{'} - b_3^{'} b_1) b_2 \equiv 0 \,,$$

and thus the sum (100) is equal to

$$\sum_{m=k} [...] = -(\mathcal{U}_1(u) + 4\alpha(u)) \sum_k a_k b_k W_k \bar{W}_k \,. \tag{101}$$

The 'non-diagonal' part of the sum (99) can be transformed to

$$\sum_{m \neq k} [...] = -(\mathcal{U}_1(u) + 4\alpha(u)) \sum_{m \neq k} a_m b_k W_m \bar{W}_k \tag{102}$$

by employing for $m \neq k$ the evident identities

$$a_m^{''} W_k - a_m^{'} W_k^{'} + a_m W_k^{''} = -\mathcal{U}_1(u)\, a_m(u)\, W_k \,, \tag{103}$$

which can easily be checked. Collecting the diagonal and non-diagonal parts and returning to X_1 and X_2, we find that the $\bar{C}_u$ constraint (97) is equivalent to

$$(\mathcal{U}_1(u) + 4\alpha(u)) X_1 X_2 = 0 \,. \tag{104}$$

Repeating the same derivation for C_v and recalling that $X_1 X_2 \neq 0$, we complete the proof of Eqs.(95), with the evident notation $C_\mu(u) \equiv 4\alpha(u)$, $C_\nu(u) \equiv 4\beta(v)$.

Note that in our derivation the number of the Liouville components may be arbitrary. On the other hand, we essentially used the simple structure of the $\mathcal{A}_2$ solutions and thus our derivation cannot be directly applied to other the higher $\mathcal{A}_N$ equations.

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